

Worst case analysis of random circuits using Taylor models and Bernstein polynomials techniques

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Abstract

In this paper, a novel paradigm to optimize the Taylor models based worst-case analysis results of random circuits responses in time domain has been proposed. The methodology leverages the so-called Bernstein polynomials to yield a conservative, yet tight, prediction of the worst case bounds. This proposed framework has been used and verified in the time-domain analysis of an exemplary linear circuit, which demonstrated its feasibility and strength.

Keywords—worst case analysis; Taylor models; Bernstein polynomials; reliability; uncertainty

Introduction

Circuits and system variability has long been impacting the stability and reliability of electrical and electronic equipment, and only in the last century or so have scientists and engineers been able to investigate the responses of electronic devices affected by unknown uncertainties via numerical techniques [1]. This pursuit was made possible through advancements in computer techniques [2-3]. Among these numerical methods, Taylor models (TM), as a sophisticated approach combining the force of the traditional interval analysis (IA) method, offers reasonable inherent worst case (WC) responses of circuits and systems in a timely fashion [4-6]. In the TM operations, all random parameters are represented in terms of truncated Taylor expansions and an interval variable, while the interval variable accounts for the truncation errors. The corresponding TM operations abide both polynomial and IA rules. The final range of the response is provided by both the polynomial and the IA-remainder, hence it is indispensable to determine the exact bound of this truncated polynomial, which is indeed non-trivial, especially for high-order multivariate polynomials [6]. A rough approximation of the polynomial bound will possibly lead to unnecessarily strict margins, which are very unlikely to occur [3].

In order to improve the TM based simulation results, one idea is to adopt more sophisticated bound functions for the polynomial part. Bernstein polynomials play an extremal position in some classes of operations [7]. It is capable of determining somehow tight bounds of a multivariate polynomial over a closed box, and is therefore widely used for global optimization [8] and numerical approximation [9]. The

aim of this paper is to outline how Bernstein basis helps improve the TM-based WC analysis of passive linear circuits in time domain.

Taylor Models

The idea of the so called TM approach is to represent a nonlinear function as Taylor polynomials combining an interval remainder. For instance, given f of an interval variable $\bar{x} = [a, b]$ (with the center of \bar{x} being $x_0 = (a + b) / 2$), The TM form of $f(\bar{x})$ around x_0 should be

$$f(\bar{x}) = P_f(x - x_0) + I_f \quad (1)$$

where

$$P_f(x - x_0) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \quad (2)$$

is the expanded polynomial part of $f(x)$ around x_0 , and I_f being an interval value, which is commonly defined as the Lagrange remainder of (2), the readers are suggested to see [4-5] for detailed discussion on the calculation of IA-remainder.

Denote $B(\cdot)$ as the operator to calculate the bound of a function, and the overall bound of the TM form function can be represented as

$$B(f(\bar{x})) = B(P_f) + I_f \quad (3)$$

This overall bound is obtained from the IA-sum of two sub-intervals, and it should enclose $f(x)$ between two curves for $\forall x \in [a, b]$. Ideally, $I_f = [0, 0]$ implies that $P_f(x)$ provides an exact parametric representation of $f(x)$, however in reality, the TM form bound of (3) always provides over-estimated results.

Given two TM form functions $f(\bar{x}) = P_f(x) + I_f$ and $g(\bar{x}) = P_g(x) + I_g$, we can readily write the algebraic operations between them, for instance, the product between them yields

$$\begin{aligned} u(\bar{x}) &= f(\bar{x}) g(\bar{x}) = (P_f(x) + I_f)(P_g(x) + I_g) \\ &= P_f(x)P_g(x) + P_f(x)I_g + P_g(x)I_f + I_fI_g \end{aligned} \quad (4)$$

The first term in (4) can be further expressed as

$$P_f(x) P_g(x) = P_u(x) + P_e(x) \quad (5)$$

where the order of $P_u(x)$ is equal to that of $P_f(x)$ and $P_g(x)$, thus accounting for the polynomial part of the new TM form of $u(x)$, whilst $P_e(x)$ accounts for the higher-order contribution. In this sense, the new remainder of the product includes four remaining terms and should be calculated as follows.

$$I_u = B(P_e) + B(P_f)I_g + B(P_g)I_f + I_f I_g \quad (6)$$

It is worth noting that the use of Taylor models requires a fast and accurate bound operator $B(\cdot)$ of the polynomials. Since IA has been included in each step of the TM operations, a rough bound of the polynomial part may lead to a large over-estimation in a long chain of realistic operations. In the univariate case, the bounds of a polynomial can be readily obtained with analytical approaches, while in the multivariate case it is non-trivial and requires suitable solutions. One possible solution is introduced in the next part.

Multivariate Polynomial Bounds

As has been stated in the previous section, at the end of a series of TM operations, we have obtained a TM form function, the polynomial part of which is commonly represented by the so called *power basis*

$$P(x) = \sum_{i=0}^l a_i x^i, \quad x^i = \prod_{\mu=1}^n x_{\mu}^{i_{\mu}} \quad (7)$$

in x_1, \dots, x_n , of the degree $l = (l_1, \dots, l_n)$, as well as a box

$$X = [\underline{x}_1, \bar{x}_1] \times \dots \times [\underline{x}_n, \bar{x}_n] \quad (8)$$

This is only one of an infinite number of bases for the space of polynomials. In order to obtain a tight outer bound for $P(x)$ over X , one method is to convert the power basis form multivariate polynomial (7) into the Bernstein form, which reads:

$$P(x) = \sum_{i=0}^l b_i B_i(x) \quad (9)$$

where

$$B_i(x) = \binom{l}{i} x^i (1-x)^{l-i} \quad (10)$$

form a basis, which constitutes the vector space of polynomials with degree l , and

$$b_i = \sum_{j=0}^i \binom{j}{i} a_j, \quad 0 \leq i \leq l \quad (11)$$

are the so-called Bernstein coefficients. It is relevant to point out that the evaluation of b_i in (11) is effective only for *unit* box, namely $X = [0,1]^n$. For any arbitrary non-empty box, the general form of Bernstein coefficients can be found in (12)

$$b_i = \sum_{j=0}^i \binom{j}{l} (\bar{x} - \underline{x})^j \sum_{k=j}^l \binom{k}{j} x^{k-j} a_k \quad (12)$$

Once we obtain the coefficients set b_i of the Bernstein expansion for a given polynomial, the range of the polynomial over a particular box is determined simultaneously, which is tightly bounded by these coefficients, as the property states: the range of a polynomial P over a box X is contained between the minimum value and the maximum value of the Bernstein coefficients [6].

$$\min_i \{b_i\} \leq P(x) \leq \max_i \{b_i\}, \quad x \in X \quad (13)$$

Obviously, one has to compute all of the Bernstein coefficients to determine the minimum and maximum, and this can be extremely time-consuming in high-order operations consisting of a relatively large number of variables. Smith proposed a new approach to calculate the B-coefficients of multivariate polynomials in [6], where he begins with multivariate monomials. Consider the following polynomial with only one term

$$q(x) = a_k x^k, \quad x = (x_1, \dots, x_n), \quad 0 \leq k \leq l \quad (14)$$

The Bernstein coefficients for monomial can be modified as

$$b_i = a_k \sum_{j=0}^{\min(i,k)} \binom{j}{l} (\bar{x} - \underline{x})^j \binom{k}{j} x^{k-j} \quad (15)$$

On condition that a polynomial P is composed of m terms, the overall Bernstein coefficients can be acquired from the sum of all the coefficients of every single monomial. In this case, the computational complexity is simply linear to the number of total polynomials.

Numerical Results

In this section, we have collected the application results of the proposed technique to the time domain simulation of the passive circuits depicted in Fig. 1, where all component values R, L , and C are defined by an interval value with given bounds.

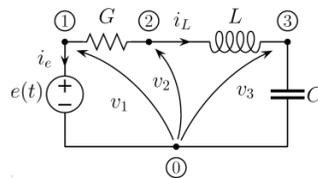


Fig. 1. An exemplary example of RLC circuit used to validate the proposed TM method and Bernstein technique for worst-case analysis of random circuits in time domain.

The nominal value of the parameters in Fig. 1 is $R_0 = 1\Omega$, $L_0 = 1H$, and $C_0 = 1F$. with all variables a $\pm 10\%$ tolerance. The circuit equation in form of matrix writes as follows:

$$\mathbf{Y}_d \frac{d}{dt} \mathbf{w}(t) + \mathbf{Y}_g \mathbf{w}(t) = \mathbf{J} \quad (16)$$

where $\mathbf{w} = [v_1(t), v_2(t), v_3(t), i_L(t), i_e(t)]^T$ is the vector of unknown variables, $\mathbf{J} = [0, 0, 0, 0, e(t)]^T$, and

$$\mathbf{Y}_g = \begin{bmatrix} G & -G & 0 & 0 & 1 \\ -G & G & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix}, \quad \mathbf{Y}_d = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -L \end{bmatrix}$$

are the interval valued parameter matrices. To solve the differential equation (16), we resort to the custom Trapezoidal rule, namely, if $\frac{dw}{dt} = f(t, w)$, then

$$\frac{w_{n+1} - w_n}{h} = \frac{1}{2} [f(t_n, w_n) + f(t_n, w_{n+1})] \quad (17)$$

where $h = t_{n+1} - t_n$ is the time step. The final time-domain analysis using sampled-data iterative map is performed as

$$(2\mathbf{Y}_d + h\mathbf{Y}_g) \mathbf{w}_{n+1} = (2\mathbf{Y}_d - h\mathbf{Y}_g) \mathbf{w}_n + h(\mathbf{a}_{n+1} + \mathbf{a}_n) \quad (18)$$

Unfortunately, a divergence is observed for the time-domain responses, as can be seen in Fig. 2, due to the unlimited growth of the IA-remainder.

If and only if, we use a higher-order Bernstein bound method, the over-estimation is beat successfully, and the optimized responses of capacitance voltage for the Fig. 1 circuit can be observed as shown in Fig. 3.

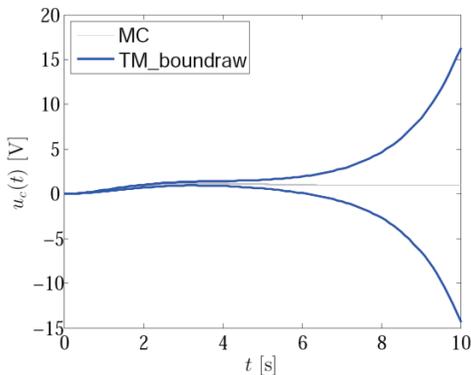


Fig. 2. Error explosion due to the divergence of IA-remainder

Conclusion

In this work, we have illustrated a Taylor models algorithm framework combing the strength of Bernstein polynomials to jointly obtain the time-domain responses of a random circuit

with bounded uncertain parameters. Simulated results demonstrated both the accuracy and the feasibility of the technique, and highlighted that the proposed framework allows for achieving a substantial improvement compared with plain TM calculations.

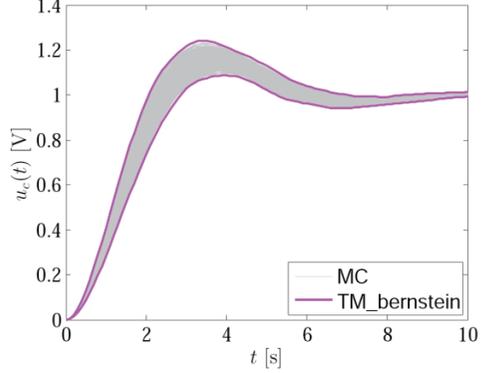


Fig. 3. Time-domain solution of the capacitor voltage. The upper bound and lower bound are obtained by TMs combining Bernstein bound solution (magenta lines), and are compared with 10,000 Monte Carlo samples (gray area).

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